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Lorentzian Lattices and E-Polytopes

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Abstract: We consider certain E_n-type root lattices embedded within the standard Lorentzian lattice $\mathbb{Z}^{n+1}$ $(3 \leq n \leq 8)$ and study their discrete geometry from the point of view of del Pezzo surface geometry. The lattice $\mathbb{Z}^{n+1}$ decomposes as a disjoint union of affine hyperplanes which satisfy a certain periodicity. We introduce the notions of line vectors, rational conic vectors, and rational cubics vectors and their relations to E-polytopes. We also discuss the relation between these special vectors and the combinatorics of the Gosset polytopes of type $(n-4)_{21}$.

Keywords: lorentzian lattice; weyl group; root lattice; dual lattice; lines; gosset polytope; E-polytope

MSC: 03G10; 05B25; 52B20; 14J26

1. Introduction

Lattices and their related discrete geometry appear naturally in the study of algebraic surfaces. In this context, the Picard group Pic(S) of a del Pezzo surface S is a typical example of Lorentzian lattice, determining, in turn, a root lattice with Weyl group W of E-type [1–7]. Special divisor classes in Pic S, such as lines, rulings and exceptional systems are of interest [5–7]. The convex hull of the set of lines is a Gosset polytope and some special divisor classes correspond to the facets of these Gosset polytopes [3,5]. Such divisor classes were studied in [6]. The present article builds on this work.

Consider the Lorentzian lattice $\mathbb{Z}^{n+1}$ $(3 \leq n \leq 8)$ with signature $(1, n)$, identified geometrically with $H^2(S, \mathbb{Z})$ where S is a general del Pezzo surface of degree $9-n$. This lattice carries a canonical element K_n with length $(K_n, K_n) = 9-n$. The orthogonal sub-lattice $K_n^{\perp}$ turns out to be a root lattice of E_n-type, in the terminology of [3]. We study the affine lattice hyperplanes Λ_n^a consisting of lattice elements D with $(D, K_n) = a$. We prove an identification between Λ_n^a and elements of the discriminant group $\left(K_n^{\perp}\right)^{\vee}/K_n^{\perp}$. This fact is used to explain a periodicity appearing in the lattice structure on Λ_n^a.

Motivated by considerations in [6], we introduce the notions of lines, rational conic vectors and rational cubics as elements in $\mathbb{Z}^{n+1} \otimes \mathbb{Q}$. These special elements form orbits under the action of the Weyl group W_n and their lattice structures are and naturally related to the ones of certain E_n-polytopes, such as the Gosset polytopes $(n-4)_{21}$, $2_{(n-4)1}$, $1_{(n-4)2}$. We compute the total numbers of these subsets via theta series associated with root lattices and their duals.

We also consider lattice elements that can be written as sum of lines and study the configurations of lines analog to the discrete geometry of the Gosset polytopes. As an application, we show that each root d in $K_n^{\perp}$ can be written as a difference of two distinct perpendicular lines.

In the next article, we will consider certain type of K3 surfaces related to del Pezzo surfaces via the involutive automorphism. The lattice structures of the K3 surfaces are also Lorentzian ones with Weyl action, and play key roles to understand the geometry of K3 surfaces. We expect the study of E-polytopes of del Pezzo surfaces can be extended to the discrete geometry of K3 surfaces.

2. Root Lattices and Hyperplanes

Let $\mathbb{Z}^{n+1}$ be the Lorentzian lattice with rank $n+1$ and standard basis $\{e_i \mid 0 \leq i \leq n\}$ satisfying

$$(e_0, e_0) = 1,\ (e_i, e_i) = -1 \text{ for } i = 1, ..., n,\ (e_i, e_j) = 0 \text{ for } i \neq j\ .$$

We denote by $(,)$ the lattice inner product. We shall refer to (v, v) as the *length* of the lattice point v.

Motivated by the geometry of del Pezzo surfaces in algebraic geometry, we consider the following specific integral vector

$$K_n := -3e_0 + e_1 + ... + e_n$$

which we shall refer to as the *canonical vector*. We also restrict the range of the parameter n to $3 \leq n \leq 8$, so that K_n has positive length $(K_n, K_n) = 9 - n$.

Let $K_n^{\perp} \subset \mathbb{Z}^{n+1}$ be the orthogonal complement sub-lattice

$$K_n^{\perp} := \left\{ D \in \mathbb{Z}^{n+1} \,\middle|\, (D, K_n) = 0 \right\}$$

It follows that the restricted inner product (.) on $K_n^{\perp}$ is negative definite and $K_n^{\perp}$ is in fact a root lattice (see [3,4]) with a root system given by:

$$\mathbf{R}_n := \left\{ D \in \mathbb{Z}^{n+1} \mid (D, D) = -2,\ (D, K_n) = 0 \right\}\ .$$

The overall number of roots is as given below:

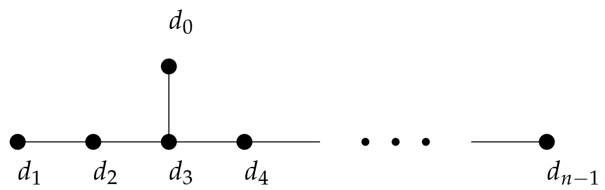
n	3	4	5	6	7	8
$\lvert\mathbf{R}_n\rvert$	12	20	40	72	126	240

Total numbers of the roots of $\mathbf{R}_n$

A set of simple roots in $\mathbf{R}_n$ is constructed as:

$$d_0 = e_0 - e_1 - e_2 - e_3, \quad d_i = e_i - e_{i+1},\ 1 \leq i \leq n-1,$$

with an associated Dynkin diagram of E_n-type (see [3,5,6]):



Dynkin diagram of E_n $n \geq 3$

We shall therefore refer to the list below as the *extended list of* E_n's:

n	3	4	5	6	7	8
E_n	$A_1 \times A_2$	A_4	D_5	E_6	E_7	E_8

Again borrowing terminology from algebraic geometry, we shall refer to the product

$$\deg(D) = (D, -K_n)$$

as *the degree* of a vector D. The length and degree of a vector D in $\mathbb{Z}^{n+1}$ satisfy the inequality

$$(D, D)\,(K_n, K_n) \leq (D, -K_n)^2 = \deg(D)^2$$

This follows from the Lemma below (the backwards Cauchy-Schwartz in Lorentzian setting).

Lemma 1. *Let v in $\mathbb{Z}^{n+1}$ with positive length. Then any D in $\mathbb{Z}^{n+1}$ satisfies,*

$$(D,D)\,(v,v) \leq (D,v)^2$$

The equality holds if and only if D and v are scalar multiple of each other.

Proof. Since v has positive length, every point of its orthogonal complement $\{v\}^{\perp} \subset \mathbb{Z}^{n+1} \otimes \mathbb{R}$ has non-positive length. Hence:

$$\left(D - \frac{(D,v)}{(v,v)} v,\; D - \frac{(D,v)}{(v,v)} v \right) \leq 0\,.$$

The needed inequality follows. □

Next, we introduce the fixed-degree hyperplanes $\Lambda_n^r \subset \mathbb{Z}^{n+1}$ defined as

$$\Lambda_n^r := \left\{ D \in \mathbb{Z}^{n+1} \;\middle|\; (D, -K_n) = r \right\}.$$

In particular, Λ_n^0 is the root lattice $K_n^{\perp}$.
Consider furthermore the subsets M_n^r of $\mathbb{Z}^{n+1} \otimes \mathbb{Q}$ defined by

$$M_n^r := \frac{r}{(K_n, K_n)}\, K_n \;+ \Lambda_n^r\,.$$

Lemma 2. *The following statements hold: (1) M_n^r is a subset of the dual lattice $\left(K_n^{\perp}\right)^{\vee}$, (2) For any $v \in M_n^r$, one has*

$$M_n^r \;=\; v + K_n^{\perp} \;=\; \{\, v + \gamma \mid \gamma \in K_n^{\perp} \,\}\,.$$

Proof. (1) Note that, for each u in Λ_n^r, we have

$$\left(\frac{r}{(K_n, K_n)}\, K_n + u,\; K_n \right) = 0\,.$$

Hence, M_n^r is a subset in $\Lambda_n^0 \otimes \mathbb{Q}$. Furthermore, for each $w \in \Lambda_n^0$, we have

$$\left(\frac{r}{(K_n, K_n)}\, K_n + u,\; w \right) = (v, w) \in \mathbb{Z}$$

One concludes that M_n^r is a subset of the dual lattice $\left(K_n^{\perp}\right)^{\vee}$.

(2) Note that, for any two v_1 and v_2 in M_n^r, $v_1 - v_2 \in K_n^{\perp}$. Thus, for any choice $v \in M_n^r$, we get $M_n^r \subset v + K_n^{\perp}$.

Conversely, for any γ in $K_n^{\perp}$, one verifies:

$$\left(v + \gamma - \frac{r}{(K_n, K_n)}\, K_n,\; -K_n \right) = -\,(v, K_n) + 0 + r = r\,.$$

Moreover:

$$v + \gamma - \frac{r}{(K_n, K_n)}\, K_n \;=\; \left(v - \frac{r}{(K_n, K_n)}\, K_n \right) + \gamma \in \mathbb{Z}^{n+1}\,.$$

Hence, one has $v + K_n^{\perp} \subset M_n^r$. One concludes $M_n^r = v + K_n^{\perp}$. □

We note that Lemma 2 above provides a canonical group morphism:

$$\varphi\colon \mathbb{Z} \to D(K_n^{\perp})$$

which associates to any fixed-degree subset Λ_n^r a coset $v + K_n^{\perp}$. Here,

$$D(K_n^{\perp}) = \left(K_n^{\perp}\right)^{\vee} / K_n^{\perp}$$

denotes the discriminant group of the negative-definite root lattice $K_n^{\perp}$, which is known to be a cyclic group of order $9 - n$. One can easily see that the morphism φ is surjective.

Let us also note that the choice of v in M_n^r representing the coset may be selected in a canonical way. Consider the basis of simple roots $\mathcal{B} = \{ d_i \in \mathbf{R}_n \mid i = 1, ..., n \}$ of $K_n^{\perp}$, as introduced earlier. Then:

$$[\left(K_n^{\perp}\right)^{\vee} : K_n^{\perp}] = \det(Gram) = (K_n, K_n) = 9 - n$$

where $Gram$ is the $n \times n$ symmetric matrix whose (i, j) entry is given by the pairing (d_i, d_j) of the corresponding simple roots. Consider, in addition, the following concept, as introduced, for instance, in [8], Section 4:

Definition 1. *The fundamental parallelepiped associated with $\mathcal{B}$ is, by definition:*

$$\Pi(\mathcal{B}) := \left\{ \sum_{i=1}^{n} a_i d_i \,\middle|\, a_i \in [0,1] \text{ for each } i \right\} \subset K_n^{\perp} \otimes \mathbb{R} \subset \left(K_n^{\perp}\right)^{\vee} \otimes \mathbb{R}$$

It follows then (see Lemma 4.2 of [8]) that for each $w \in \left(K_n^{\perp}\right)^{\vee}$ one has a unique decomposition

$$w = u_o + v_o$$

where $u_o \in \Pi(\mathcal{B}) \cap \left(K_n^{\perp}\right)^{\vee}$ and $v_o \in K_n^{\perp}$. We then have the following:

Theorem 1. *For each $r \in \mathbb{Z}$, there exists a unique $u_o \in \Pi(\mathcal{B}) \cap \left(K_n^{\perp}\right)^{\vee}$ such that $M_n^r = u_o + K_n^{\perp}$.*

Proof. We know that each element w in M_n^r has a unique decomposition

$$w = u_o + v_o$$

with $u_o \in \Pi(\mathcal{B}) \cap \left(K_n^{\perp}\right)^{\vee}$ and $v_o \in K_n^{\perp}$. In fact, the element u_o is independent of the choice of w. Indeed, let w and w' be two elements in M_n^r with decompositions:

$$w = u_o + v_o, \quad w' = u_o' + v_o'$$

as above. Then $w - w' \in K_n^{\perp}$ and

$$w - w' = (u_o - u_o') + (v_o - v_o') \,.$$

By the uniqueness of the decomposition, it follows that $u_o - u_o' = 0$. Hence $u_o = u_o'$. □

Remark 1. *Note that the above provides then a one-to-one correspondence between $\Pi(\mathcal{B}) \cap \left(K_n^{\perp}\right)^{\vee}$ and the classes of the discriminant group $D(K_n^{\perp}) \simeq \mathbb{Z}/(9-n)\mathbb{Z}$. In particular:*

$$\left| \Pi(\mathcal{B}) \cap \left(K_n^{\perp}\right)^{\vee} \right| = |D(K_n^{\perp})| = \det(Gram) = 9 - n.$$

The above considerations, in connection with Theorem 1, uncover to the following periodic feature of the lattice hyperplanes Λ_n^r.

Corollary 1. *Let $r_1, r_2 \in \mathbb{Z}$ with $r_1 \equiv r_2 \pmod{9-n}$. Then $M_n^{r_1} = M_n^{r_2}$. In particular, one has a canonical one-to-one (translation) correspondence between $\Lambda_n^{r_1}$ and $\Lambda_n^{r_2}$.*

We may conclude therefore that the disjoint union

$$\mathbb{Z}^{n+1} = \bigcup_{r \in \mathbb{Z}} \Lambda_n^r$$

carries a periodicity modulo $9-n$. For instance, for each n between 3 and 8, the hyperplane Λ_n^{9-n} is the translation of the root lattice $K_n^{\perp}$ by the lattice point $-K_n$.

3. Fundamental Lattice Vectors

As already mentioned in the previous section, the action of the Weyl group W_n preserves the length and degree of a vector D in $\mathbb{Z}^{n+1}$. Therefore, W_n acts on each given subset consisting of integral vectors in $\mathbb{Z}^{n+1}$ with fixed length and degree.

Following up again on ideas from algebraic geometry, we shall refer to the quantity:

$$\frac{1}{2}\left[\, (D, D) - \deg(D) \,\right] + 1 \;=\; \frac{1}{2}\left[\, (D, D) + (D, K_n) \,\right] + 1$$

as the arithmetic genus of the integral vector D. In particular, by a slight abuse of terminology, we shall refer to integral vectors D satisfying

$$\deg(D) = (D, D) + 2$$

as *rational*. In what follows, we shall study the sets of rational vectors of low positive degrees: 1, 2 and 3. Certain relations connecting these objects to the theory of semiregular polytopes (as studied in [3,5–7]) will be discussed.

Note that the set of the rational vectors D with $\deg D = 0$ is precisely $\mathbf{R}_n$—the set of roots in $K_n^{\perp}$.

3.1. Lines

We shall refer to vectors $l \in \mathbb{Z}^{n+1}$ satisfying $\deg(l) = 1$ and $(l, l) = -1$ as *lines*. The set of all lines:

$$\mathbf{L}_n := \left\{ l \in \mathbb{Z}^{n+1} \mid (l, l) = (l, K_n) = -1 \right\}.$$

is finite and lies within the hyperplane Λ_n^1. As discussed in the previous section, one has:

$$\Lambda_n^1 \;=\; \left(\frac{1}{n-9} \right) K_n + M_n^1$$

where M_n^1 is a $K_n^{\perp}$-coset in $\left(K_n^{\perp}\right)^{\vee}$. One obtains:

$$\mathbf{L}_n \;=\; \left\{ \left(\frac{1}{n-9} \right) K_n + \gamma \mid \gamma \in M_n^1 \text{ with } (\gamma, \gamma) = -\frac{10-n}{9-n} \right\}.$$

One can determine then the size of the set $\mathbf{L}_n$ via the standard arguments of Conway-Slone [9]. The relevant quantity is the coefficient of degree

$$1 + \frac{1}{(K_n, K_n)} \;=\; \frac{10-n}{9-n}$$

in theta series associated with the dual lattice $\left(K_n^{\perp}\right)^{\vee}$ of the root lattice $K_n^{\perp}$ (which is of E_n-type).

For instance, the lines in $\mathbf{L}_7$ appear as:

$$-\frac{1}{2}K_7 + \gamma$$

where $\gamma \in M_7^1$ with

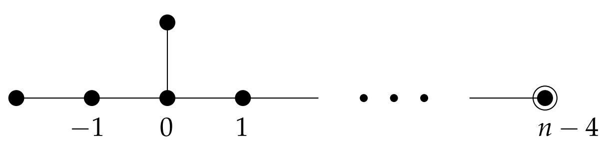
$$(\gamma, \gamma) = \frac{3}{2}\,.$$

The coefficient of degree $3/2$ in the theta series of the dual lattice of E_7 is 56 (see [5,9]) and hence, $|\mathbf{L}_7| = 56$. Similar arguments led one to the following list:

n	3	4	5	6	7	8
$\lvert\mathbf{L}_n\rvert$	6	10	16	27	56	240

Total numbers of the lines

We note that the above $|\mathbf{L}_n|$ list matches with a combinatorial count of a different nature—the number of vertices of Gosset polytopes $(n-4)_{21}$. These are certain n-dimensional $(n = 3,4,5,6,7,8)$ semiregular polytopes discovered by Gosset [10,11]. The Coxeter groups of $(n-4)_{21}$ are known to be of E_n-type, with associated Coxeter-Dynkin diagram given as follows.



Coxeter-Dynkin diagram of n_{21} $n \neq 3$

Note that the vertex figure of $(n-4)_{21}$ is $(n-5)_{21}$. For $n \neq 3$, the facets of the $(n-4)_{21}$-polytope consist of regular simplexes α_{n-1} and crosspolytopes β_{n-1}, but all the lower dimensional subpolytopes are regular simplexes. Coxeter referred to 4_{21}, 3_{21} and 2_{21} as Gosset polytopes but the Gosset polytope list may be expanded according to our E_n list. Note that the Gosset polytope $(-1)_{21}$ has an isosceles (non-equilateral) triangle as the vertex figure (see [5]).

Connecting with the Conway-Slone theta arguments, one observes (see [5]) that the subset $\mathbf{L}_n$ is acted upon transitively by the Weyl group W_n. Via the Weyl action, one constructs then a Gosset polytope $(n-4)_{21}$ in Λ_n^1 as a convex hull of $\mathbf{L}_n$ in $\Lambda_n^1 \otimes \mathbb{Q}$. One obtains:

Theorem 2 (Theorem 4.2 in [5])**.** *The lines of* $\mathbf{L}_n$ *correspond bijectively to the vertices of a Gosset polytope* $(n-4)_{21}$ *in* $\Lambda_n^1 \otimes \mathbb{Q}$.

Remark 2. *Note, for instance, that, by Corollary 1,* Λ_8^1 *is a translation of the root lattice* $\Lambda_8^0 = K_8^\perp$*. The lines in* $\mathbf{L}_8$ *in* Λ_8^1 *are then bijectively matched to the set of roots* $\mathbf{R}_8$ *in* Λ_8^0*. The root polytope (convex hull of* $\mathbf{R}_8$ *in* $K_8^\perp \otimes \mathbb{Q}$*) is then the Gosset polytope* 4_{21}.

3.2. Rational Conic Vectors

We shall refer to rational vectors $a \in \mathbb{Z}^{n+1}$ with $\deg(a) = 2$ as *rational conics*. In the context of Del Pezzo surfaces in algebraic geometry, these lattice vectors are associated with rulings. We shall denote their set here by:

$$\mathbf{Con}_n := \left\{ a \in \mathbb{Z}^{n+1} \mid (a,a) = 0,\ (a, K_n) = -2 \right\}.$$

$\mathbf{Con}_n$ are finite sets. As with the previous discussion, $|\mathbf{Con}_n|$ may be read via the Conway-Sloane argument ([5,9]), from the degree $4/(9-n)$ coefficient of the appropriate theta series of the dual $E_n^\vee$

lattice. For example, if we consider $n = 8$, the root lattice $K_8^\perp$ is E_8 and hence self-dual. The appropriate theta series is then:

$$\Theta_{K_8^\perp} = \sum_{m=0}^{\infty} N_m q^m, \; N_m = 240\sigma_3(\frac{m}{2})$$

where $\sigma_r(m) = \sum_{d|m} d^r$. The elements in $\mathbf{Con}_8$ correspond to the lattice points of self-pairing $4/(9-8) = 4$ and hence $|\mathbf{Con}_n|$ is the coefficient of q^4, namely $240(1+2^3) = 2160$. In a similar manner, one computes all $|\mathbf{Con}_n|$ for $3 \leq n \leq 8$ as follows:

n	3	4	5	6	7	8
$\lvert\mathbf{Con}_n\rvert$	3	5	10	27	126	2160

Total numbers of the rational conics

The rational conic vectors form a single orbit under the action of the Weyl group W_n. They also correspond to the $(n-1)$-crosspolytopes of the Gosset polytope $(n-4)_{21}$, which form one of the two possible types of facets on $(n-4)_{21}$. We refer the reader to [5] for the details.

We also note that for each conic vector a can be written as a formal sum:

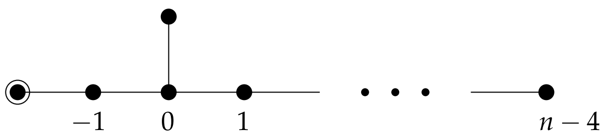
$$a = l_1^a + l_2^a.$$

where l_1^a, l_2^a are two lines satisfying $(l_1^a, l_2^a) = 1$. In terms of the crosspolytope interpretation, the two lines l_1^a, l_2^a correspond to two antipodal vertices of the $(n-1)$-crosspolytope associated with the conic vector a. Since there are precisely $(n-1)$ pairs of antipodal vertices in a given $(n-1)$-crosspolytope, it follows that, for each conic vector a, one concludes (see [5]) that there are precisely $(n-1)$ pairs of lines l_1^a, l_2^a as above.

Let us also recall the following result:

Lemma 3 (Ref. [5] Lemma 5.6)**.** *Let a and l be a rational conic vector and a line in $\mathbb{Z}^{n+1}$, respectively. Then, one has: (1) The line l corresponds to a vertex of the $(n-1)$-crosspolytope associated with a if and only if $(l, a) = 0$. (2) Assume $a = l_1^a + l_2^a$ where l_1^a and l_2^a are lines. Then the line l corresponds to a vertex of the $(n-1)$-crosspolytope associated with a if and only if $(l, l_1^a) = (l, l_2^a) = 0$.*

We also note that $\mathbf{Con}_n$ is bijectively related to the set of vertices of the polytope $2_{(n-4)1}$. The polytopes $2_{(n-4)1}$ $(n = 3, 4, 5, 6, 7, 8)$ are n-dimensional semiregular polytopes whose Coxeter groups are E_n, constructed as follows:



Coxeter-Dynkin diagram of $2_{(n-4)1}$ $n \neq 3$

The vertex figure of $2_{(n-4)1}$ is an $(n-1)$-demicube. Moreover, assuming $n \neq 3$, the facets of $2_{(n-4)1}$ are regular either simplexes $\alpha_{(n-1)}$ or semiregular polytopes of type $2_{(n-5)1}$. It follows then (see [7]) that the convex hull of $\mathbf{Con}_n$ in the hyperplane $\Lambda_n^2 \otimes \mathbb{Q}$ is $2_{(n-4)1}$.

Remark 3. *Note that, by Corollary 1 in the $n = 7$ context, the hyperplane Λ_7^2 is in an one-to-one correspondence with to root lattice $\Lambda_7^0 = K_7^\perp$. Under this mapping, the set of rational conics $\mathbf{Con}_7$ in Λ_7^2 corresponds (via $a \mapsto a + K_7$) to the set of roots $\mathbf{R}_7$. The root polytope (convex hull of $\mathbf{R}_7$ in $K_7^\perp \otimes \mathbb{Q}$) is then 2_{31}, as more generally stated earlier.*

3.3. Rational Cubic Vectors

We shall refer to rational vectors b in $\mathbb{Z}^{n+1}$ with $\deg b = 3$ as *rational cubic* vectors. The set of all such rational cubics

$$\mathbf{Cub}_n := \left\{ b \in \mathbb{Z}^{n+1} \mid (b,b) = 1,\ (b, K_n) = -3 \right\}$$

is a finite sets and its cardinal $|\mathbf{Cub}_n|$ may again be determined via the Conway-Sloane procedure [5,9] by finding the coefficient of degree

$$-1 + \frac{9}{(K_n, K_n)} = \frac{9}{9-n}$$

in the theta series of dual $E_n^\vee$ lattice. One obtains:

n	3	4	5	6	7	8
$\lvert\mathbf{Cub}_n\rvert$	2	5	15	72	576	17520

Total numbers of the rational cubics

The following Lemma of [5] establishes a combinatorial relationship between rational cubic vectors and configurations of n mutually orthogonal lines.

Lemma 4 (Ref. [5] Theorem 5.3)**.** *Let b be a rational cubic vector in $\mathbb{Z}^{n+1}$. One has: (1) If $3 \leq n \leq 7$, then $3b + K_n$ can be written as a sum of n lines $l_1, l_2, \cdots l_n$ with $(l_i, l_j) = 0$ for $i \neq j$. Conversely, for each configuration $l_1, l_2, \cdots l_n$ of mutually orthogonal lines,*

$$\frac{l_1 + ... + l_n - K_n}{3}$$

is a rational cubic vector in $\mathbb{Z}^{n+1}$. (2) Let $n = 8$. Given any choice of eight mutually orthogonal lines $l_1, ..., l_8$, one has that:

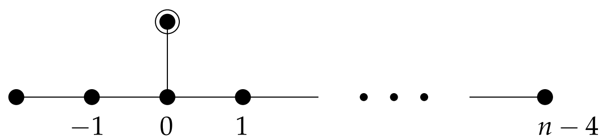
$$\frac{l_1 + ... + l_8 - K_8}{3}$$

is a rational cubic vector. (3) Let $n = 8$. The vector $3b + K_8$ is a sum of eight mutually orthogonal lines if and only if $(b + 3K_8)/2$ is not integral (i.e., not a root in $K_8^\perp$).

Based on the above, we see that for $3 \leq n \leq 7$, the set $\mathbf{Cub}_n$ forms an orbit of the Weyl group action. Moreover, elements of $\mathbf{Cub}_n$ ($n \neq 8$) correspond bijectively to $(n-1)$-simplexes in the Gosset polytope $(n-4)_{21}$. The case $n = 8$ is special. Now $\mathbf{Cub}_8$ partitions into two W_8-orbits. One orbit is in bijective correspondence with the root set in $\mathbf{R}_8$. The other orbit corresponds to the set of 7-simplexes in the Gosset polytope 4_{21}. One has:

$$|\mathbf{Cub}_8| = 17520 = 240 + 17280 = |\mathbf{R}_8| + |\{\text{7-simplexes in } 4_{21}\}| \,.$$

Let us also note that, for $3 \leq n \leq 7$, the elements of $\mathbf{Cub}_n$ are in bijective correspondence with the vertices of the $1_{(n-4)2}$ polytope. These n-dimensional polytopes are convex and semiregular. Their symmetry groups are the Coxeter groups E_n, and can be constructed as in the following diagram:



Coxeter-Dynkin diagram of $1_{(n-4)2}$ $n{\neq}3$

Note that the vertex figure of $1_{(n-4)2}$ is a birectified n-simplex. For $n \neq 3$, the facets of $1_{(n-4)2}$ are either semiregular polytopes $1_{(n-5)2}$ or $(n-1)$-demicubes.

We also note that, assuming $n \neq 8$, the convex hull of $\mathbf{Cub}_n$ in $\Lambda_n^3 \otimes \mathbb{Q}$ is $1_{(n-4)2}$. We refer the reader to [7], for the details.

A particular situation worth mentioning is also the case of rational cubics of $n = 6$. In this case, via Corollary 1, Λ_6^3 is in bijective correspondence $(v \mapsto v + K_6)$ with the root lattice $\Lambda_6^0 = K_6^{\perp}$. Under this correspondence, $\mathbf{Cub}_6$ maps to the set of roots $\mathbf{R}_6$. As predicted above, the root polytope (convex hull of $\mathbf{R}_6$ in $K_6^{\perp} \otimes \mathbb{Q}$) is 1_{22}.

4. Line Configurations

In this section, we study lattice vectors D in $\mathbb{Z}^{n+1}$ that can be written as formal sums of lines:

$$D = l_1 + ... + \ l_k \text{ where } l_1, ..., l_k \ \in \mathbf{L}_n$$

As noted in earlier works [5–7,12] by the second author, if one pre-sets the lattice pairings between the lines $l_1, ..., l_k$, the set of possible $D's$ and the set of possible line configurations carry deep and interesting symmetries involving the Weyl groups W_n and the discrete subpolytope geometry of Gosset polytopes $(n-4)_{21}$.

4.1. Lattice Pairings of Lines

Let us collect a few facts pertaining to lattice pairings associated with line vectors. These will be applied to a study of root configurations in the next section. Note that certain aspects of these facts concerning the geometry of del Pezzo surfaces may be found in [5,6].

We begin by noting that, given two lines l_1 and l_2 in $\mathbb{Z}^{n+1}$, one has:

$$-1 \leq (l_1, l_2) \leq \frac{2}{(K_n, K_n)} + 1 = \frac{11-n}{9-n}$$

This follows via applying Lemma 1 successively to vectors $l_1 + l_2$ and $l_1 - l_2$, respectively. We also observe that one can have $(l_1, l_2) = -1$ if and only if $l_1 = l_2$, as in this situation $l_1 - l_2$ is a vector in $K_n^{\perp}$ of null self-pairing.

The case of $(l_1, l_2) = 0$ has interesting combinatorial interpretations. In this situation, the lines l_1, l_2 correspond to a pair of vertices joined by an edge in the Gosset polytope $(n-4)_{21}$. This fact may be seen via considerations in Theorem 2. Following this line of thought, we shall refer such an unordered pair $\{l_1, l_2\}$ as an *edge*. The set of all edges:

$$A_n := \{\{l_1, l_2\} \mid l_1, l_2 \in \mathbf{L}_n \ , (l_1, l_2) = 0\}.$$

has then a cardinal given as follows:

n	3	4	5	6	7	8
$\lvert A_n \rvert$	9	30	80	216	756	6720

Total numbers of the edges in Gosset polytopes $(n-4)_{21}$

Moreover, as discussed in [5], for any edge $\{l_1, l_2\}$ the lattice vector $l_1 + l_2$ gives the edge barycenter and the barycenter set

$$\tilde{A}_n := \{l_1 + l_2 \mid l_1, l_2 \in \mathbf{L}_n, \ (l_1, l_2) = 0\}$$

is in one-to-one correspondence with A_n. The elements in $\tilde{A}_n$ are lattice vectors D in $\mathbb{Z}^{n+1}$ satisfying

$$(D, D) = -2 \text{ and } (D, K_n) = -2,$$

and, in fact, one can argue (see [5]) that these are the only vectors satisfying this pair of conditions.

$$\tilde{A}_n = \left\{ D \in \mathbb{Z}^{n+1} \mid (D,D) = -2, \ (D,K_n) = -2 \right\}$$

The set $\tilde{A}_n$ also forms a full orbit under the Weyl group W_n action ([5]).

As a side note, let us also mention that in fact, if one considers lattice vector D in $\mathbb{Z}^{n+1}$ satisfying

$$(D,D) = -3, \ (D,K_n) = -3 \, ,$$

it can be proved (see [5]) that there exists a unique triple of lines l_a, l_b and l_c satisfying:

$$D = l_a + l_b + l_c, \ \ (l_a,l_b) = (l_a,l_c) = (l_b,l_c) = 0 \, .$$

The above are some typical examples of lattice vectors associated with configurations of lines. These cases are particularly nice as the associated configurations of lines turn out to be unique. In general, this feature is not to be expected.

4.2. Line Hierarchy and the Gosset Polytope $(n-4)_{21}$

Let l be a line in $\mathbf{L}_n$. As discussed earlier, l corresponds to a vertex of the Gosset polytope $(n-4)_{21}$. For $k \in \mathbb{Z}$, we define then:

$$B_k^n\,(l) =: \left\{ l' \in \mathbf{L}_n \mid (l',l) = k \right\} .$$

We have the following cases:

(1) $k = -1$. Then as observed earlier:

$$B_{-1}^n\,(l) = \{l\}$$

(2) $k = 0$. The elements of $B_0^n\,(l)$ correspond then to the edges of $(n-4)_{21}$ originating at l. This is called *the vertex figure* and can be identified with vertices of a Gosset polytope $(n-5)_{21}$. Hence, one obtains an interesting correspondence between the sets $B_0^n\,(l)$ and $\mathbf{L}_{n-1}$. In particular:

n	3	4	5	6	7	8
$\lvert B_0^n\,(l)\rvert$	3	6	10	16	27	56

(3) $k = 1$. Note that for $l' \in B_1^n\,(l)$, one has that $l + l' \in \mathbf{Con}_n$. Similarly, any $l' \in B_1^n\,(l)$ may be obtained by subtracting l from a rational conic vector. We obtain therefore a bijective identification between elements of $B_1^n\,(l)$ and the set of rational conic vectors $\mathbf{Con}_n$. In particular, in the light of previous considerations, every element of $l' \in B_1^n\,(l)$ corresponds to a $(n-1)$-crosspolytope containing l as a vertex. The two vertices corresponding to l and l' are antipodal in the $(n-1)$-crosspolytope. We obtain a list for $\lvert B_1^n\,(l)\rvert$ as follows.

n	3	4	5	6	7	8
$\lvert B_1^n\,(l)\rvert$	2	3	5	10	27	126

(4) $k = 2$. This case only appears when $n = 7$ or 8. For $n = 7$, one obtains:

$$B_2^7\,(l) = \{-K_7 - l\} \ ,$$

whereas, for $n = 8$:

$$B_2^8\,(l) = \left\{ -2K_8 - l' \mid l' \in B_0^n\,(l) \right\} .$$

In particular, one has $\lvert B_2^8\,(l)\rvert = 56$.

(5) $k = 3$ The set $B_3^n(l)$ is non-empty only when $n = 8$. In this situation:

$$B_3^8(l) = \{-2K_8 + l\}$$

4.3. Roots as Configurations of Orthogonal Lines

In this section, we show that the roots in $\mathbf{R}_n$ can be seen as a difference of orthogonal lines.

Theorem 3. *Let $d \in \mathbf{R}_n$ be a root. Then, there exists a unique (ordered) pair of two lines l_1, $l_2 \in \mathbf{L}_n$ such that:*

$$d = l_1 - l_2, \quad (l_1, l_2) = 0 \ .$$

Proof. Consider the set

$$\mathcal{E}_n := \{ \ l_1 - l_2 \ | \ l_1, l_2 \in \mathbf{L}_n, \ (l_1, l_2) = 0 \ \}$$

One clearly has $\mathcal{E}_n \subset \mathbf{R}_n$.

The two sets $\mathbf{R}_n$ and $\mathcal{E}_n$ are finite. In what follows, we shall count the elements in $\mathcal{E}_n$ and show that $|\mathbf{R}_n| = |\mathcal{E}_n|$. This fact then implies $\mathcal{E}_n = \mathbf{R}_n$ and hence the statement of the Theorem follows.

Note that each edge $\{l_1, l_2\} \in A_n$ produces two elements in $\mathcal{E}_n$. However, there could potentially be multiple pairs of orthogonal lines returning the same root as difference. From the point of view of the Gosset polytopes $(n-4)_{21}$ geometry, two parallel edges produce the same root, up to a sign. Below, we perform a count of all the possible edges parallel to a given fixed edge.

The case $n = 3$ can be treated via straightforward verification. We have $|\mathbf{L}_3| = 6$ with:

$$\mathbf{L}_3 \ = \ \{e_1, \ e_2, \ e_3, \ e_0 - e_1 - e_2, \ e_0 - e_1 - e_3, \ e_0 - e_2 - e_3\}$$

which partitions into two triples of mutually orthogonal lines:

$$\{e_1, \ e_2, \ e_3\} \ \text{ and } \ \{e_0 - e_1 - e_2, \ e_0 - e_1 - e_3, \ e_0 - e_2 - e_3\} \ .$$

The orthogonal differences produce then all the twelve roots in $\mathbf{R}_3$.

For $4 \leq n \leq 8$, we shall use a previous observation (see Section 4.1)—the Weyl group W_n acts transitively on the set of edges of the polytope $(n-4)_{21}$. We shall then choose $\{e_n, e_{n-1}\}$ as the fixed edge.

Consider $4 \leq n \leq 6$. In this situation, the pairing between lines is at most one. Any edge $\{l_1, l_2\}$ parallel to $\{e_n, e_{n-1}\}$ defines then a rational conic vector $m = e_n + l_2 = e_{n-1} + l_1$ in $(n-4)_{21}$ which, in turn, corresponds to a $(n-1)$-crosspolytope for $(n-4)_{21}$ with e_n, e_{n-1} antipodal vertices. The rational conic vector m conversely determines the parallel edge $\{l_1, l_2\}$. By Lemma 3 the set of rational conic vectors containing the edge $\{e_n, e_{n-1}\}$ is then in bijective correspondence to the set of lines $l \in \mathbf{L}_n$ satisfying $(l, e_n) = 1$ and $(l, e_{n-1}) = 0$. The relevant number for us to compute is then the number of elements in $B_1^n(e_n) \cap B_0^n(e_{n-1})$. In each case we then obtain:

(1) $n = 4$. We have $\left|B_1^4(e_4) \cap B_0^4(e_3)\right| = 2$. Thus:

$$|\mathcal{E}_4| = 2 \times \frac{|A_4|}{1 + \left|B_1^4(e_4) \cap B_0^4(e_3)\right|} = 2 \times \frac{30}{3} = 20 = |\mathbf{R}_4|$$

(2) $n = 5$. We have $\left|B_1^5(e_5) \cap B_0^5(e_4)\right| = 3$. Therefore,

$$|\mathcal{E}_5| = 2 \times \frac{|A_5|}{1 + \left|B_1^5(e_5) \cap B_0^5(e_4)\right|} = 2 \times \frac{80}{4} = 40 = |\mathbf{R}_5|$$

(3) $n = 6$. We have $\left|B_1^6(e_6) \cap B_0^6(e_5)\right| = 5$. Hence:

$$|\mathcal{E}_6| = 2 \times \frac{|A_6|}{1 + \left|B_1^6(e_6) \cap B_0^6(e_5)\right|} = 2 \times \frac{216}{6} = 72 = |\mathbf{R}_6|$$

The remaining situations are $n = 7$ and 8. In these cases, the lattice pairing between lines could reach 2 and 3. Hence, there will more parallel edges to $\{e_n, e_{n-1}\}$ besides those associated with vertices in the common $(n-1)$-crosspolytopes.

(4) $n = 7$. In this case we have $\left|B_1^7(e_7) \cap B_0^7(e_6)\right| = 10$ and $\left|B_2^7(e_7) \cap B_1^7(e_6)\right| = 1$. We also note that:

$$B_2^7(e_7) \cap B_1^7(e_6) = \{-K_7 - e_7\}$$

with the corresponding edge $\{-K_7 - e_7, -K_7 - e_6\}$. Therefore:

$$|\mathcal{E}_7| = 2 \times \frac{|A_7|}{1 + \left|B_1^7(e_7) \cap B_0^7(e_6)\right| + \left|B_2^7(e_7) \cap B_1^7(e_6)\right|} = $$
$$= 2 \times \frac{756}{12} = 126 = |\mathbf{R}_7|$$

(5) $n = 8$. We have:

$$\left|B_1^8(e_8) \cap B_0^8(e_7)\right| = 27, \quad \left|B_2^8(e_8) \cap B_1^8(e_7)\right| = 27, \quad \left|B_3^8(e_8) \cap B_2^8(e_7)\right| = 1\,.$$

Moreover, note that the sets $B_1^8(e_8) \cap B_0^8(e_7)$ and $B_2^8(e_8) \cap B_1^8(e_7)$ are bijectively related via the map $l \longmapsto -(2K_8 + l)$. In addition, we have $B_3^8(e_8) \cap B_2^8(e_7) = \{-2K_8 - e_8\}$. We compute:

$$|\mathcal{E}_8| = 2 \times \frac{|A_8|}{1 + \left|B_1^8(e_8) \cap B_0^8(e_7)\right| + \left|B_2^8(e_8) \cap B_1^8(e_7)\right| + \left|B_3^8(e_8) \cap B_2^8(e_7)\right|}$$
$$= 2 \times \frac{6720}{56} = 240 = |\mathbf{R}_8|\,.$$

This completes the proof of the Theorem. □

Remark 4. *Note that, in case $n = 7$, the involution $G\colon \mathbf{L}_7 \mapsto \mathbf{L}_7$ with $G(l) := -K_7 - l$ is known in the literature as the Gieser transform. Similarly, in the case $n = 8$, the involution $B\colon \mathbf{L}_8 \mapsto \mathbf{L}_8$ given by $B(l) = -2K_8 - l$ is known as the Bertini transform. These isometries act naturally on the Gosset polytopes 3_{21} and 4_{21} respectively ([5]).*

Let us also include the following application of Theorem 3:

Corollary 2. *Let $l \in \mathbf{L}_n$ be a line. There exists then an ordered set of mutually perpendicular lines $\{l_1, ..., l_n\}$ containing l and a rational cubic b such that*

$$d_0 = b - l_1 - l_2 - l_3,\ d_i = l_i - l_{i+1},\ 1 \le i \le n-1,$$

forms a set of simple roots for the lattice $K_n^{\perp}$ (which has E_n-type).

Proof. Let $l \in \mathbf{L}_n$. it is easy to see there are ordered perpendicular lines containing l. We denote one of them as $l_1, ..., l_n (= l)$. By applying Lemma 3 we find a rational cubic b as

$$b = \frac{l_1 + ... + l_n - K_n}{3}.$$

Then b satisfies $b \cdot l_i = 0$ for each i, and $(b - l_1 - l_2 - l_3) \cdot (l_i - l_{i+1}) = 0$ for each $i \neq 3$. Thus we conclude

$$d_0 = b - l_1 - l_2 - l_3,\ d_i = l_i - l_{i+1},\ 1 \leq i \leq n-1,$$

are simple roots of E_n root. This gives the Corollary. □

Remark 5. *In the geometry of del Pezzo Surfaces, the lines discussed here play key roles in the cohomology of blow-up and blow-down transformations. For a fixed line $l \in \mathbf{L}_n$ the "blow-down" of $\mathbf{L}_n$ via l can be viewed as $l^{\perp} \cap \mathbf{L}_n$, set that can be naturally identified with $\mathbf{L}_{n-1}$. Since elements of $\mathbf{L}_n$ is corresponded to the vertices of Gosset polytope $(n-4)_{21}$, the identification*

$$l^{\perp} \cap \mathbf{L}_n = \mathbf{L}_{n-1}$$

is equivalent to the fact that the vertex figure of the Gosset polytope $(n-4)_{21}$ gives the Gosset polytope $(n-5)_{21}$. This interesting interplay between the Del Pezzo surface geometry and the combinatorics of the associated Gosset polytopes will be discussed in a subsequent work.

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