

See discussions, stats, and author profiles for this publication at: <https://www.researchgate.net/publication/261100690>

On the Sums of Any k Points in Finite Fields

Article in SIAM Journal on Discrete Mathematics · March 2014

DOI: 10.1137/140982441 · Source: arXiv

CITATIONS

3

READS

30

3 authors:



David John Covert

University of Missouri - St. Louis

11 PUBLICATIONS 114 CITATIONS

SEE PROFILE



Doowon Koh

Chungbuk National University

41 PUBLICATIONS 381 CITATIONS

SEE PROFILE



Youngjin Pi

Chungbuk National University

4 PUBLICATIONS 7 CITATIONS

SEE PROFILE

ON THE SUMS OF ANY k POINTS IN FINITE FIELDS

DAVID COVERT, DOOWON KOH*, AND YOUNGJIN PI

ABSTRACT. For a set $E \subset \mathbb{F}_q^d$, we define the k -resultant magnitude set as $\Delta_k(E) = \{\|x^1 + \cdots + x^k\| \in \mathbb{F}_q : x^1, \dots, x^k \in E\}$, where $\|\alpha\| = \alpha_1^2 + \cdots + \alpha_d^2$ for $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{F}_q^d$. In this paper we find a connection between a lower bound of the cardinality of the k -resultant magnitude set and the restriction theorem for spheres in finite fields. As a consequence, it is shown that if $E \subset \mathbb{F}_q^d$ with $|E| \geq Cq^{\frac{d+1}{2} - \frac{1}{5d+2}}$, then $|\Delta_3(E)| \geq cq$ for $d = 4$ or $d = 6$, and $|\Delta_k(E)| \geq cq$ for even dimensions $d \geq 4$ and $k \geq 4$. In addition, we prove that if $d \geq 8$ is even, and $|E| \geq C_\varepsilon q^{\frac{d+1}{2} - \frac{1}{9d-18} + \varepsilon}$ for $\varepsilon > 0$, then $|\Delta_3(E)| \geq cq$ for some $0 < c < 1$.

1. INTRODUCTION

Let \mathbb{F}_q^d , $d \geq 2$, be the d -dimensional vector space over a finite field with q elements. Throughout the paper, we assume that the characteristic of \mathbb{F}_q is not equal to two. For $E \subset \mathbb{F}_q^d$, the distance set, denoted by $\Delta_2(E)$, is defined by

$$\Delta_2(E) = \{\|x - y\| \in \mathbb{F}_q : x, y \in E\},$$

where $\|\alpha\| = \alpha_1^2 + \cdots + \alpha_d^2$ for $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{F}_q^d$. The Erdős-Falconer distance problem in the finite field setting asks for the minimal threshold β such that if $|E| \geq Cq^\beta$ for a sufficiently large constant C , then we have $|\Delta_2(E)| \geq cq$ for some $0 < c \leq 1$. The first distance result was obtained by Bourgain, Katz, and Tao ([1]) when $q \equiv 3 \pmod{4}$ is a prime. Iosevich and Rudnev ([12]) studied the general field case, and they obtained the first explicit exponents. Using discrete Fourier machinery, they demonstrated that if $E \subset \mathbb{F}_q^d$ with $|E| \geq Cq^{(d+1)/2}$, for a sufficiently large constant C , then $|\Delta_2(E)| = q$.

The authors in [6] constructed arithmetic examples which show that the exponent $(d+1)/2$ due to Iosevich and Rudnev is sharp at least in odd dimensions. Thus, the Erdős-Falconer distance problem has been completely resolved in odd dimensions. On the other hand, it has been conjectured for even dimensions $d \geq 2$ that the exponent $(d+1)/2$ can be improved to the exponent $d/2$. While this conjecture is open for all even dimensions, the sharp exponent $(d+1)/2$ for odd dimensions was improved for dimension two by the authors in [2]. More precisely they proved that if $E \subset \mathbb{F}_q^2$ with $|E| \geq Cq^{4/3}$ for a sufficiently large constant C ,

Date: March 26, 2014.

2010 *Mathematics Subject Classification.* 52C10, 42B05, 11T23.

*Corresponding Author

Key words and phrases: Erdős distance problem, restriction theorems, finite fields.

Doowon Koh was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology(2012R1A1A1001510) .

then $|\Delta_2(E)| \geq cq$ for some $0 < c < 1$. However, the exponent $(d+1)/2$ has not been improved for higher even dimensions $d \geq 4$. For further discussion on distance problems in finite fields, readers may refer to [5, 7, 13, 14, 15, 16, 22, 23]. See also [4, 3], and references contained therein for recent results on the distance problems in the ring setting.

The Erdős-Falconer distance problem in finite fields can be extended in various directions. One such direction is as follows. For each integer $k \geq 2$, let us consider a function $M_k : (\mathbb{F}_q^d)^k \rightarrow \mathbb{F}_q$. Given this function, determine the minimal value β such that whenever $E \subset \mathbb{F}_q^d$ satisfies $|E| \geq Cq^\beta$ for a sufficiently large constant C , we have $|M_k(E^k)| \geq cq$ for some constant $0 < c \leq 1$ independent of q . Note that when $M_2(x, y) = \|x - y\|$ for $x, y \in \mathbb{F}_q^d$, we are reduced the Erdős-Falconer distance problem in the finite field setting as

$$\Delta_2(E) = M_2(E \times E) = \{\|x - y\| \in \mathbb{F}_q : x, y \in E\}.$$

For $k \geq 2$, we will study the function

$$M_k(x^1, x^2, \dots, x^k) = \|x^1 \pm x^2 \pm \dots \pm x^k\| \quad \text{for } x^s \in \mathbb{F}_q^d, s = 1, 2, \dots, k,$$

and we denote $M_k(E^k)$ by $\Delta_k(E)$ for $E \subset \mathbb{F}_q^d$. Namely, for $E \subset \mathbb{F}_q^d$, we define

$$\Delta_k(E) = \{\|x^1 \pm x^2 \pm \dots \pm x^k\| \in \mathbb{F}_q : x^s \in E, s = 1, 2, \dots, k\}.$$

As the choice of signs will be independent of our results, we shall simply define

$$\Delta_k(E) = \{\|x^1 + x^2 + \dots + x^k\| \in \mathbb{F}_q : x^s \in E, s = 1, 2, \dots, k\}.$$

Throughout the paper, the set $\Delta_k(E)$ will be referred to as the k -resultant magnitude set. For brevity, we call $\Delta_2(E)$ the distance set, and when $k = 3$, we simply call $\Delta_3(E)$ the magnitude set.

Question 1.1. *Let $E \subset \mathbb{F}_q^d$, $d \geq 2$, and $k \geq 2$ be an integer. Determine the smallest $\beta > 0$ such that if $|E| \geq Cq^\beta$ with a sufficiently large constant $C > 1$, then $|\Delta_k(E)| \geq cq$ for some $0 < c \leq 1$.*

It is clear that $|\Delta_{k_1}(E)| \leq |\Delta_{k_2}(E)|$ for $2 \leq k_1 \leq k_2$. Therefore, as k becomes larger, one might expect the smaller β as the answer to Question 1.1. However, we conjecture that the answer to Question 1.1 is independent of k . For example, if $q = p^2$ for prime p and $E = \mathbb{F}_p^d$, then it clearly follows that $|E| = q^{d/2}$ and $|\Delta_k(E)| = \sqrt{q}$ for all $k \geq 2$. This example says that β in Question 1.1 cannot be smaller than $d/2$ which is the conjectured exponent for the Erdős-Falconer distance problem in even dimensions. This leads us to the following conjecture.

Conjecture 1.2. *Let $E \subset \mathbb{F}_q^d$. If $d \geq 2$ is even and $|E| \geq Cq^{d/2}$ for a sufficiently large constant C , then for every integer $k \geq 2$, there exists a constant $0 < c \leq 1$ such that*

$$|\Delta_k(E)| \geq cq.$$

1.1. Statement of results. The techniques used by Iosevich and Rudnev in [12] show that if $|E| \geq Cq^{\frac{d+1}{2}}$ for a sufficiently large constant C , then $\Delta_k(E) = \mathbb{F}_q$. Note that the counterexamples for the Erdős-Falconer distance problem immediately show that the exponent $(d+1)/2$ cannot be improved in general for odd dimensions. Thus, we shall only focus on investigating the size of $\Delta_k(E)$ where $E \subset \mathbb{F}_q^d$ is a subset of an even dimensional vector space. In this paper we demonstrate that the

exponent $(d + 1)/2$ for the magnitude set can be improved for even $d \geq 4$. More precisely, we have the following results.

Theorem 1.3. *Let $E \subset \mathbb{F}_q^d$. Suppose that C is a sufficiently large constant.*

- (1) *If $d = 4$ or 6 , and $|E| \geq Cq^{\frac{d+1}{2} - \frac{1}{6d+2}}$, then $|\Delta_3(E)| \geq cq$ for some $0 < c \leq 1$.*
- (2) *If $d \geq 4$ is even, $k \geq 4$ is an integer, and $|E| \geq Cq^{\frac{d+1}{2} - \frac{1}{6d+2}}$, then $|\Delta_k(E)| \geq cq$ for some $0 < c \leq 1$.*

Theorem 1.4. *Suppose that $d \geq 8$ is even and $E \subset \mathbb{F}_q^d$. Then given $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that if $|E| \geq C_\varepsilon q^{\frac{d+1}{2} - \frac{1}{9d-18} + \varepsilon}$, then $|\Delta_3(E)| \geq cq$ for some $0 < c \leq 1$.*

It seems from our results that the exponent $(d + 1)/2$ can be improved for the Erdős-Falconer distance problem in even dimensions $d \geq 4$.

Remark 1.5. Aside from thinking of the cardinality of $\Delta_3(E)$ as the number of distinct distances of any three vectors in $E \subset \mathbb{F}_q^d$, we can also consider it as the number of distinct distances between the origin and the centers of mass of triangles determined by $E \subset \mathbb{F}_q^d$ if q has characteristic greater than 3. To see this, notice that if $x, y, z \in \mathbb{F}_q^d$, then $(x + y + z)/3$ can be considered as the center of mass of the triangle with vertices x, y, z .

1.2. Outline of the paper. In the remaining parts of the paper, we first provide preliminary lemmas in Section 2. In Section 3, we obtain the necessary restriction estimates for spheres. In the final section, we deduce the formula for $|\Delta_k(E)|$ and we provide the link between the set $\Delta_k(E)$ and the restriction estimates for spheres.

2. DISCRETE FOURIER ANALYSIS AND RELATED LEMMAS

As a main technical tool, discrete Fourier analysis plays an important role in proving our results. In this section, we review the basic definitions, and we collect preliminary lemmas which are essential for providing a lower bound for $|\Delta_k(E)|$.

2.1. Discrete Fourier analysis. Throughout this paper, χ denotes a nontrivial additive character of \mathbb{F}_q . The choice of the character χ will be independent of the results in this paper. The orthogonality of the character χ implies

$$\sum_{x \in \mathbb{F}_q^d} \chi(m \cdot x) = \begin{cases} 0 & \text{if } m \neq (0, \dots, 0) \\ q^d & \text{if } m = (0, \dots, 0), \end{cases}$$

where $m \cdot x$ denotes the usual dot-product. Given a function $g : \mathbb{F}_q^d \rightarrow \mathbb{C}$, the Fourier transform of g , denoted by \tilde{g} , is defined as

$$\tilde{g}(x) = \sum_{m \in \mathbb{F}_q^d} g(m) \chi(-x \cdot m) \quad \text{for } x \in \mathbb{F}_q^d.$$

On the other hand, if $f : \mathbb{F}_q^d \rightarrow \mathbb{C}$, then we denote by \hat{f} the **normalized** Fourier transform of the function f . Thus, we have

$$\hat{f}(m) = \frac{1}{q^d} \sum_{x \in \mathbb{F}_q^d} f(x) \chi(-x \cdot m) \quad \text{for } m \in \mathbb{F}_q^d.$$

We also write $f^\vee(m)$ for $\widehat{f}(-m)$. Notice that $(\widehat{f^\vee})(x) = f(x)$ for $x \in \mathbb{F}_q^d$. Namely, the Fourier inversion theorem in this content is given by the formula

$$f(x) = \sum_{m \in \mathbb{F}_q^d} \widehat{f}(m) \chi(m \cdot x) \quad \text{for } x \in \mathbb{F}_q^d.$$

As a direct application of the orthogonality relation of χ , it follows that

$$\sum_{m \in \mathbb{F}_q^d} |\widehat{f}(m)|^2 = \frac{1}{q^d} \sum_{x \in \mathbb{F}_q^d} |f(x)|^2.$$

We refer to this formula as the Plancherel theorem. As simple consequence from the Plancherel theorem, it follows that if $E \subset \mathbb{F}_q^d$, then

$$\sum_{m \in \mathbb{F}_q^d} |\widehat{E}(m)|^2 = \frac{|E|}{q^d}.$$

Here, throughout this paper, by abuse of notations, we identify the set $E \subset \mathbb{F}_q^d$ with the characteristic function on the set E . Since $|\widehat{E}(m)| \leq q^{-d}|E|$, it is trivial that for every integer $k \geq 2$,

$$(2.1) \quad \sum_{m \in \mathbb{F}_q^d} |\widehat{E}(m)|^k \leq \frac{|E|^{k-2}}{q^{d(k-2)}} \sum_{m \in \mathbb{F}_q^d} |\widehat{E}(m)|^2 = \frac{|E|^{k-1}}{q^{dk-d}}.$$

We now collect information about the normalized Fourier transform on the sphere. For $t \in \mathbb{F}_q$, the sphere $S_t \subset \mathbb{F}_q^d$ is defined by

$$S_t = \{x \in \mathbb{F}_q^d : x_1^2 + \cdots + x_d^2 = t\}.$$

It is well known from Theorem 6.26 and Theorem 6.27 in [17] that if $d \geq 3$ and $t \in \mathbb{F}_q$, then

$$(2.2) \quad |S_t| = q^{d-1}(1 + o(1)).$$

The following result follows immediately from Lemma 4 in [10].

Proposition 2.1. *Let $d \geq 2$ be even and $t \in \mathbb{F}_q$. Then, for $m \in \mathbb{F}_q^d$,*

$$\widehat{S}_t(m) = q^{-1} \delta_0(m) + q^{-d-1} G^d \sum_{\ell \in \mathbb{F}_q^*} \chi\left(t\ell + \frac{\|m\|}{4\ell}\right),$$

where $\delta_0(m)$ is the delta-function, so that $\delta_0(m) = 1$ for $m = (0, \dots, 0)$ and $\delta_0(m) = 0$ otherwise, and G denotes the Gauss sum

$$G = \sum_{s \in \mathbb{F}_q^*} \eta(s) \chi(s),$$

where η is the quadratic character of \mathbb{F}_q , and $\mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\}$. In particular, we have

$$\widehat{S}_0(m) = q^{-1} \delta_0(m) + q^{-d-1} G^d \sum_{\ell \in \mathbb{F}_q^*} \chi(\|m\|\ell) \quad \text{for } m \in \mathbb{F}_q^d.$$

Remark 2.2. Recall that the Gauss sum satisfies $|G| = \sqrt{q}$. For $a, b \in \mathbb{F}_q$, the Kloosterman sum is defined by

$$K(a, b) := \sum_{\ell \in \mathbb{F}_q^*} \chi(a\ell + b/\ell).$$

It is well known that $|K(a, b)| \leq 2\sqrt{q}$ for $ab \neq 0$. For the proof of the Gauss and Kloosterman sum estimation, see [11, 17].

The following result was proved in Proposition 2.2 in [16].

Proposition 2.3. *For $m, \alpha \in \mathbb{F}_q^d$, we have*

$$\sum_{t \in \mathbb{F}_q} \widehat{S}_t(m) \overline{\widehat{S}_t(\alpha)} = q^{-1} \delta_0(m) \delta_0(\alpha) + q^{-d-1} \sum_{s \in \mathbb{F}_q^*} \chi(s(\|m\| - \|\alpha\|)).$$

2.2. Evaluation of the counting function ν_k . Let $E \subset \mathbb{F}_q^d$ and let $k \geq 2$ be an integer. For $t \in \mathbb{F}_q$, we define the counting function $\nu_k(t)$ by

$$\begin{aligned} \nu_k(t) &:= |\{(x^1, x^2, \dots, x^k) \in E^k : \|x^1 + x^2 + \dots + x^k\| = t\}| \\ &= \sum_{x^1, \dots, x^k \in E} S_t(x^1 + x^2 + \dots + x^k). \end{aligned}$$

Applying the Fourier inversion theorem to $S_t(x^1 + x^2 + \dots + x^k)$, it follows from the definition of the normalized Fourier transform that

$$(2.3) \quad \nu_k(t) = q^{dk} \sum_{m \in \mathbb{F}_q^d} \widehat{S}_t(m) \left(\overline{\widehat{E}(m)} \right)^k.$$

Then an L^2 estimate of ν_k is as follows.

Lemma 2.4. *Let $E \subset \mathbb{F}_q^d$, $d \geq 2$. Then we have*

$$\sum_{t \in \mathbb{F}_q} \nu_k^2(t) \leq q^{-1} |E|^{2k} + q^{2dk-d} \sum_{r \in \mathbb{F}_q} \left| \sum_{\alpha \in S_r} \left(\widehat{E}(\alpha) \right)^k \right|^2.$$

Proof. Since $\nu_k^2(t) = \nu_k(t) \overline{\nu_k(t)}$, we see from (2.3) that

$$\sum_{t \in \mathbb{F}_q} \nu_k^2(t) = q^{2dk} \sum_{m, \alpha \in \mathbb{F}_q^d} \left(\overline{\widehat{E}(m)} \right)^k \left(\widehat{E}(\alpha) \right)^k \left(\sum_{t \in \mathbb{F}_q} \widehat{S}_t(m) \overline{\widehat{S}_t(\alpha)} \right).$$

From Proposition 2.3, we conclude that

$$\begin{aligned} \sum_{t \in \mathbb{F}_q} \nu_k^2(t) &= q^{-1} |E|^{2k} + q^{2dk-d} \sum_{\substack{m, \alpha \in \mathbb{F}_q^d: \\ \|m\| = \|\alpha\|}} \left(\overline{\widehat{E}(m)} \right)^k \left(\widehat{E}(\alpha) \right)^k - q^{2dk-d-1} \left| \sum_{\alpha \in \mathbb{F}_q^d} \left(\widehat{E}(\alpha) \right)^k \right|^2 \\ &\leq q^{-1} |E|^{2k} + q^{2dk-d} \sum_{\substack{m, \alpha \in \mathbb{F}_q^d: \\ \|m\| = \|\alpha\|}} \left(\overline{\widehat{E}(m)} \right)^k \left(\widehat{E}(\alpha) \right)^k \\ &= q^{-1} |E|^{2k} + q^{2dk-d} \sum_{r \in \mathbb{F}_q} \left| \sum_{\alpha \in S_r} \left(\widehat{E}(\alpha) \right)^k \right|^2. \end{aligned}$$

□

We need the following lemma.

Lemma 2.5. *Suppose that $d \geq 2$ even and $k \geq 2$ is an integer. If $E \subset \mathbb{F}_q^d$ with $|E| \geq 3q^{d/2}$, then we have*

$$(|E|^k - \nu_k(0))^2 \geq \frac{|E|^{2k}}{9}.$$

Proof. Combining (2.3) and Proposition 2.1, we see that

$$\begin{aligned} \nu_k(0) &= q^{dk} \sum_{m \in \mathbb{F}_q^d} \left(\widehat{E}(m) \right)^k \left(q^{-1} \delta_0(m) + q^{-d-1} G^d \sum_{\ell \in \mathbb{F}_q^*} \chi(\|m\|\ell) \right) \\ &= q^{dk-1} \left(\widehat{E}(0, \dots, 0) \right)^k + q^{dk-d-1} G^d \sum_{m \in \mathbb{F}_q^d} \left(\widehat{E}(m) \right)^k \left(\sum_{\ell \in \mathbb{F}_q^*} \chi(\|m\|\ell) \right). \end{aligned}$$

Since $\widehat{E}(0, \dots, 0) = q^{-d}|E|$, we have

$$(2.4) \quad \nu_k(0) = q^{-1}|E|^k + q^{dk-d-1} G^d \sum_{m \in \mathbb{F}_q^d} \left(\widehat{E}(m) \right)^k \left(\sum_{\ell \in \mathbb{F}_q^*} \chi(\|m\|\ell) \right).$$

Since $\nu_k(0)$ is a nonnegative real number, it is clear that

$$\nu_k(0) \leq q^{-1}|E|^k + q^{dk-d}|G|^d \sum_{m \in \mathbb{F}_q^d} |\widehat{E}(m)|^k.$$

As $|G| = q^{1/2}$, it follows from (2.1) that

$$\nu_k(0) \leq q^{-1}|E|^k + q^{d/2}|E|^{k-1}.$$

Since $q \geq 3$, this clearly implies that if $|E| \geq 3q^{d/2}$, then

$$\begin{aligned} |E|^k - \nu_k(0) &\geq |E|^k - q^{-1}|E|^k - q^{d/2}|E|^{k-1} \\ &\geq \frac{|E|^k}{3} + \left(\frac{|E|^k}{3} - q^{d/2}|E|^{k-1} \right) \geq \frac{|E|^k}{3}, \end{aligned}$$

and the statement of the lemma follows immediately. \square

We shall also use the following result.

Lemma 2.6. *Let $E \subset \mathbb{F}_q^d$. Assume that $d \geq 2$ is even and $k \geq 2$ is an integer. If $|E| \geq q^{d/2}$, then we have*

$$q^{2dk-d} \left| \sum_{m \in S_0} \left(\widehat{E}(m) \right)^k \right|^2 - \nu_k^2(0) \leq 4q^{-1}|E|^{2k}.$$

Proof. Observe from (2.4) that we can write

$$\nu_k(0) = q^{-1}|E|^k + q^{dk-d-1} G^d \sum_{m \in \mathbb{F}_q^d} \left(\widehat{E}(m) \right)^k \left(-1 + \sum_{\ell \in \mathbb{F}_q} \chi(\|m\|\ell) \right).$$

By the orthogonality relation of χ , it is easy to see that

$$\begin{aligned}\nu_k(0) &= q^{dk-d} G^d \sum_{m \in S_0} \left(\widehat{E}(m) \right)^k + \left(q^{-1} |E|^k - q^{dk-d-1} G^d \sum_{m \in \mathbb{F}_q^d} \left(\widehat{E}(m) \right)^k \right) \\ &:= A + B.\end{aligned}$$

Since $\nu_k(0) \geq 0$, it follows that

$$\begin{aligned}\nu_k^2(0) &= \nu_k(0) \overline{\nu_k(0)} = (A + B)(\overline{A} + \overline{B}) \\ &= q^{2dk-d} \left| \sum_{m \in S_0} \left(\widehat{E}(m) \right)^k \right|^2 + A\overline{B} + \overline{A}B + |B|^2.\end{aligned}$$

This observation and the definition of A and B yield that

$$\begin{aligned}q^{2dk-d} \left| \sum_{m \in S_0} \left(\widehat{E}(m) \right)^k \right|^2 - \nu_k^2(0) &\leq 2|A||B| \\ &\leq 2 \left(q^{dk-d/2} \sum_{m \in \mathbb{F}_q^d} |\widehat{E}(m)|^k \right) \left(q^{-1} |E|^k + q^{dk-d/2-1} \sum_{m \in \mathbb{F}_q^d} |\widehat{E}(m)|^k \right) \\ &\leq 2 \left(q^{d/2-1} |E|^{2k-1} + q^{d-1} |E|^{2k-2} \right),\end{aligned}$$

where (2.1) was applied to obtain the last line. We complete the proof by observing that if $|E| \geq q^{d/2}$, then

$$\max \left(q^{d/2-1} |E|^{2k-1}, q^{d-1} |E|^{2k-2} \right) \leq q^{-1} |E|^{2k}.$$

□

3. RESULTS ON THE RESTRICTION THEOREM FOR SPHERES

In this section we collect lemmas which can be obtained by applying the extension theorems for spheres in finite fields. We begin by reviewing the extension problem for spheres. We denote by (\mathbb{F}_q^d, dx) the d -dimensional vector space over \mathbb{F}_q endowed with the normalized counting measure “ dx ”. On the other hand, the dual space of (\mathbb{F}_q^d, dx) will be denoted by (\mathbb{F}_q^d, dm) where the counting measure “ dm ” is equipped. Notice that both spaces are isomorphic as an abstract group but different measures are endowed between them. For $t \in \mathbb{F}_q^*$, we consider a sphere $S_t \subset (\mathbb{F}_q^d, dx)$. For each $t \in \mathbb{F}_q^*$, we endow the sphere S_t with the normalized surface measure $d\sigma$. Recall that if $g : (\mathbb{F}_q^d, dx) \rightarrow \mathbb{C}$, then

$$\int g(x) d\sigma(x) = \frac{1}{|S_t|} \sum_{x \in S_t} g(x).$$

Also recall that if $f : (S_t, d\sigma) \rightarrow \mathbb{C}$, then the inverse Fourier transform of $f d\sigma$ is given by

$$(f d\sigma)^\vee(m) := \int f(x) \chi(m \cdot x) d\sigma(x) = \frac{1}{|S_t|} \sum_{x \in S_t} f(x) \chi(m \cdot x) \quad \text{for } m \in (\mathbb{F}_q^d, dm).$$

Since $S_t = -S_t := \{x \in \mathbb{F}_q^d : -x \in S_t\}$, the definition of the normalized Fourier transform gives

$$(3.1) \quad (d\sigma)^\vee(m) = \frac{q^d}{|S_t|} \widehat{S}_t(m) \quad \text{for } m \in (\mathbb{F}_q^d, dm).$$

With the above notation, the extension problem for the sphere S_t is to determine $1 \leq p, r \leq \infty$ such that

$$\|(fd\sigma)^\vee\|_{L^r(\mathbb{F}_q^d, dm)} \leq C \|f\|_{L^p(S_t, d\sigma)} \quad \text{for all } f : S_t \rightarrow \mathbb{C},$$

where the constant $C > 0$ is independent of the size of the underlying finite field \mathbb{F}_q . By duality, this extension estimate is the same as the following restriction estimate:

$$\|\tilde{g}\|_{L^{p'}(S_t, d\sigma)} \leq C \|g\|_{L^{r'}(\mathbb{F}_q^d, dm)} \quad \text{for all } g : \mathbb{F}_q^d \rightarrow \mathbb{C},$$

where p' and r' denote the Hölder conjugates of p and r , and \tilde{g} is the Fourier transform of g .

In the finite field setting, the extension problem for various varieties was first posed by Mockenhaupt and Tao ([21]). They mainly obtained good results for paraboloids in lower dimensions. Their results have been recently improved (see, for example, [9, 18, 19, 20]). The extension problem for spheres is more delicate than that of paraboloids, and it was studied by Iosevich and Koh. In [8], they obtained the sharp $L^2 - L^4$ extension result for circles, which the authors of [2] applied to deduce the exponent $4/3$ for the Erdős-Falconer distance problem in dimension two. Recall that if $d = 2$, then the exponent $4/3$ gives a much better result than the exponent $(d+1)/2$ which is optimal for odd dimensions. When $d \geq 3$, the $L^2 - L^{(2d+2)/(d-1)}$ extension result for spheres is also known in [8] and can be also applied to the Erdős-Falconer distance problem but we can only obtain the exponent $(d+1)/2$.

In [10], Iosevich and Koh investigated the $L^p - L^4$ spherical extension problem, and they proved the following result which improves the previous work in [8].

Proposition 3.1. *Let $d \geq 4$ be even. Then we have*

$$(3.2) \quad \|(Fd\sigma)^\vee\|_{L^4(\mathbb{F}_q^d, dm)} \lesssim \|F\|_{L^{\frac{12d-8}{9d-12}}(S_t, d\sigma)} \quad \text{for all } F \subset S_t, t \neq 0.$$

In addition, using the pigeonhole principle, (3.2) implies that

$$(3.3) \quad \|(gd\sigma)^\vee\|_{L^4(\mathbb{F}_q^d, dm)} \lesssim \|g\|_{L^{\frac{12d-8}{9d-12}}(S_t, d\sigma)} \quad \text{for all } g : S_t \rightarrow \mathbb{C}, t \neq 0.$$

Remark 3.2. Here, and throughout, we will use $X \lesssim Y$ to mean that there exists $C > 0$, independent of q such that $X \leq CY$, and we also write $Y \gtrsim X$ for $X \lesssim Y$. We use $X \sim Y$ to indicate that $\lim_{q \rightarrow \infty} X/Y = 1$. In addition, $X \lesssim Y$ means that for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that $X \leq C_\varepsilon q^\varepsilon Y$.

This proposition plays an important role in proving results for the cardinality of $\Delta_3(E)$. For the direct application to the problem, we shall invoke the following restriction lemma which can be deduced by Proposition 3.1 and the interpolation theorem.

Lemma 3.3. *Let $d \geq 4$ be even. If $k > (12d - 8)/(3d + 4) = 4 - 24/(3d + 4)$, then we have*

$$\|\tilde{f}\|_{L^k(S_t, d\sigma)} \lesssim \|f\|_{L^{\frac{k(3d+4)}{(3k-3)d+4k+2}}(\mathbb{F}_q^d, dm)} \quad \text{for all } f : \mathbb{F}_q^d \rightarrow \mathbb{C}, t \neq 0.$$

Proof. It is clear that

$$(3.4) \quad \|(gd\sigma)^\vee\|_{L^\infty(\mathbb{F}_q^d, dm)} \leq \|g\|_{L^1(S_t, d\sigma)} \quad \text{for all } g : S_t \rightarrow \mathbb{C}, t \neq 0.$$

For any even integer $d \geq 4$, it follows from (3.2) in Proposition 3.1 that

$$(3.5) \quad \|(Fd\sigma)^\vee\|_{L^4(\mathbb{F}_q^d, dm)} \lesssim \|F\|_{L^{\frac{12d-8}{9d-12}}(S_t, d\sigma)} \quad \text{for all } F \subset S_t, t \neq 0.$$

Since we assume that $k > (12d - 8)/(3d + 4)$ and $d \geq 4$, it is easy to see that $1 < k/(k-1) < (12d-8)/(9d-12)$. Therefore, using the Marcinkiewicz interpolation theorem with (3.4) and (3.5), we see that

$$\|(gd\sigma)^\vee\|_{L^{\frac{k(3d+4)}{3d-2}}(\mathbb{F}_q^d, dm)} \lesssim \|g\|_{L^{\frac{k}{k-1}}(S_t, d\sigma)} \quad \text{for all } g : S_t \rightarrow \mathbb{C}, t \neq 0.$$

Then the statement of Lemma 3.3 follows immediately from duality. \square

Observe that the hypotheses of Lemma 3.3 are satisfied if $k \geq 4$ and $d \geq 4$ is even or if $k = 3$ and $d = 4$ or 6 . However, in the case when $k = 3$ and $d \geq 8$ even, it is clear that Lemma 3.3 is not applicable. In this case, we shall alternatively use the following result.

Lemma 3.4. *Let $d \geq 8$ be an even integer. If $E \subset \mathbb{F}_q^d$ and $|E| \geq q^{(d-1)/2}$, then we have*

$$\|\tilde{E}\|_{L^3(S_t, d\sigma)} \lesssim q^{\frac{-3d^2+23d-20}{36d-96}} |E|^{\frac{15d-46}{18d-48}} \quad \text{for all } t \neq 0.$$

Proof. Let us assume for a moment that

$$(3.6) \quad \|\tilde{E}\|_{L^2(S_t, d\sigma)} \lesssim q^{\frac{-d+1}{4}} |E| \quad \text{for all } E \subset \mathbb{F}_q^d \quad \text{with } |E| \geq q^{\frac{d-1}{2}}.$$

By duality, (3.3) in Proposition 3.1 implies that

$$\|\tilde{f}\|_{L^{\frac{12d-8}{3d+4}}(S_t, d\sigma)} \lesssim \|f\|_{L^{\frac{4}{3}}(\mathbb{F}_q^d, dm)} \quad \text{for all } f : \mathbb{F}_q^d \rightarrow \mathbb{C}, t \neq 0.$$

Taking f as a characteristic function on $E \subset \mathbb{F}_q^d$, we obtain that

$$(3.7) \quad \|\tilde{E}\|_{L^{\frac{12d-8}{3d+4}}(S_t, d\sigma)} \lesssim \|E\|_{L^{\frac{4}{3}}(\mathbb{F}_q^d, dm)} = |E|^{\frac{3}{4}} \quad \text{for all } E \subset \mathbb{F}_q^d, t \neq 0.$$

Since $2 < 3 < (12d - 8)/(3d + 4)$ for $d \geq 8$, we are able to interpolate (3.6) and (3.7) for $E \subset \mathbb{F}_q^d$ with $|E| \geq q^{(d-1)/2}$. As a consequence, the conclusion of Lemma 3.4 follows and we complete the proof once we justify (3.6). Now we prove (3.6). By duality and Hölder's inequality, we see that

$$\|\tilde{E}\|_{L^2(S_t, d\sigma)}^2 \leq \|E\|_{L^1(\mathbb{F}_q^d, dm)} \|E * (d\sigma)^\vee\|_{L^\infty(\mathbb{F}_q^d, dm)} = |E| \|E * (d\sigma)^\vee\|_{L^\infty(\mathbb{F}_q^d, dm)}.$$

Let $K(m) = (d\sigma)^\vee(m) - \delta_0(m)$ for $m \in (\mathbb{F}_q^d, dm)$. Combining (3.1) with (2.2), Proposition 2.1, and Remark 2.2, we see that

$$(3.8) \quad |K(m)| \lesssim q^{\frac{-d+1}{2}} \quad \text{for all } m \in \mathbb{F}_q^d.$$

Since dm is the counting measure, it is easy to see that $\|E * \delta_0\|_{L^\infty(\mathbb{F}_q^d, dm)} = \|E\|_{L^\infty(\mathbb{F}_q^d, dm)} = 1$. Then the statement in (3.6) follows from the observation below.

$$\begin{aligned} \|\tilde{E}\|_{L^2(S_t, d\sigma)}^2 &\leq |E| \left(\|E * \delta_0\|_{L^\infty(\mathbb{F}_q^d, dm)} + \|E * K\|_{L^\infty(\mathbb{F}_q^d, dm)} \right) \\ &\leq |E| (\|E\|_{L^\infty} + \|E\|_{L^1} \|K\|_{L^\infty}) \\ &\lesssim |E| + q^{-\frac{d-1}{2}} |E|^2 \lesssim q^{-\frac{d-1}{2}} |E|^2, \end{aligned}$$

where Young's inequality was used for the second line, and where (3.8) and the assumption that $|E| \geq q^{(d-1)/2}$ was used for the last line. \square

4. PROOFS OF MAIN THEOREMS (THEOREM 1.3 AND 1.4)

We begin by deriving the formula for a lower bound of $|\Delta_k(E)|$. Let $E \subset \mathbb{F}_q^d$ and let $k \geq 2$ be an integer. For $t \in \mathbb{F}_q$, recall that the counting function $\nu_k(t)$ is defined by

$$\nu_k(t) = |\{(x^1, x^2, \dots, x^k) \in E^k : \|x^1 + x^2 + \dots + x^k\| = t\}|.$$

Also recall that the k -resultant magnitude set $\Delta_k(E)$ is given by

$$\Delta_k(E) = \{\|x^1 + x^2 + \dots + x^k\| \in \mathbb{F}_q : x^s \in E, s = 1, 2, \dots, k\}.$$

Notice that $\nu_k(t) \neq 0 \iff t \in \Delta_k(E)$. It is clear that

$$|E|^k - \nu_k(0) = \sum_{t \in \mathbb{F}_q^*} \nu_k(t).$$

Squaring both sides and using the Cauchy-Schwarz inequality, we see that

$$(|E|^k - \nu_k(0))^2 \leq |\Delta_k(E)| \sum_{t \in \mathbb{F}_q^*} \nu_k^2(t).$$

Namely, we obtain that

$$(4.1) \quad |\Delta_k(E)| \geq \frac{(|E|^k - \nu_k(0))^2}{\sum_{t \in \mathbb{F}_q^*} \nu_k^2(t)} = \frac{(|E|^k - \nu_k(0))^2}{(\sum_{t \in \mathbb{F}_q} \nu_k^2(t)) - \nu_k^2(0)}.$$

Lemma 4.1. *Let $E \subset \mathbb{F}_q^d$. Suppose that $d \geq 2$ is even and $k \geq 2$ is an integer. If $|E| \geq 3q^{d/2}$, then we have*

$$|\Delta_k(E)| \gtrsim \min \left(q, \frac{|E|^{k+1}}{q^{d-1} \left(\max_{r \in \mathbb{F}_q^*} \|\tilde{E}\|_{L^k(S_r, d\sigma)}^k \right)} \right).$$

Proof. First, we find an upper bound for $\sum_{t \in \mathbb{F}_q^*} \nu_k^2(t)$. Write

$$\sum_{t \in \mathbb{F}_q^*} \nu_k^2(t) = \left(\sum_{t \in \mathbb{F}_q} \nu_k^2(t) \right) - \nu_k^2(0).$$

From Lemma 2.4 and Lemma 2.6, we see that

$$\begin{aligned} \sum_{t \in \mathbb{F}_q^*} \nu_k^2(E) &\leq q^{-1}|E|^{2k} + q^{2dk-d} \sum_{r \in \mathbb{F}_q} \left| \sum_{\alpha \in S_r} \left(\widehat{E}(\alpha) \right)^k \right|^2 - \nu_k^2(0) \\ &\leq 5q^{-1}|E|^{2k} + q^{2dk-d} \sum_{r \in \mathbb{F}_q^*} \left| \sum_{\alpha \in S_r} \left(\widehat{E}(\alpha) \right)^k \right|^2 \\ &\lesssim q^{-1}|E|^{2k} + q^{2dk-d} \sum_{r \in \mathbb{F}_q^*} \left(\sum_{\alpha \in S_r} \left| \widehat{E}(\alpha) \right|^k \right)^2. \end{aligned}$$

Since $\widehat{E}(\alpha)$ denotes the normalized Fourier transform and $|S_r| \sim q^{d-1}$, it is easy to see that for $r \in \mathbb{F}_q^*$,

$$\left(\sum_{\alpha \in S_r} \left| \widehat{E}(\alpha) \right|^k \right)^2 = \left(\frac{|S_r|}{q^{dk}} \frac{1}{|S_r|} \sum_{\alpha \in S_r} \left| \widetilde{E}(\alpha) \right|^k \right)^2 \sim q^{2d-2-2dk} \|\widetilde{E}\|_{L^k(S_r, d\sigma)}^{2k}.$$

Combining this with the previous estimate, it follows that

$$\begin{aligned} \sum_{t \in \mathbb{F}_q^*} \nu_k^2(E) &\lesssim q^{-1}|E|^{2k} + q^{d-2} \sum_{r \in \mathbb{F}_q^*} \|\widetilde{E}\|_{L^k(S_r, d\sigma)}^{2k} \\ &\leq q^{-1}|E|^{2k} + q^{d-2} \left(\max_{r \in \mathbb{F}_q^*} \|\widetilde{E}\|_{L^k(S_r, d\sigma)}^k \right) \left(\sum_{r \in \mathbb{F}_q^*} \frac{1}{|S_r|} \sum_{x \in S_r} |\widetilde{E}(x)|^k \right) \\ &\lesssim q^{-1}|E|^{2k} + q^{-1} \left(\max_{r \in \mathbb{F}_q^*} \|\widetilde{E}\|_{L^k(S_r, d\sigma)}^k \right) \sum_{x \in \mathbb{F}_q^d} |\widetilde{E}(x)|^k \\ &= q^{-1}|E|^{2k} + q^{-1+dk} \left(\max_{r \in \mathbb{F}_q^*} \|\widetilde{E}\|_{L^k(S_r, d\sigma)}^k \right) \sum_{m \in \mathbb{F}_q^d} |\widehat{E}(m)|^k. \end{aligned}$$

Using (2.1), we obtain that

$$(4.2) \quad \sum_{t \in \mathbb{F}_q^*} \nu_k^2(E) \lesssim q^{-1}|E|^{2k} + q^{d-1}|E|^{k-1} \left(\max_{r \in \mathbb{F}_q^*} \|\widetilde{E}\|_{L^k(S_r, d\sigma)}^k \right).$$

Since it follows from Lemma 2.5 that $(|E|^k - \nu_k(0))^2 \geq \frac{|E|^{2k}}{9}$, combining (4.1) with (4.2) yields that

$$|\Delta_k(E)| \gtrsim \frac{|E|^{2k}}{q^{-1}|E|^{2k} + q^{d-1}|E|^{k-1} \left(\max_{r \in \mathbb{F}_q^*} \|\widetilde{E}\|_{L^k(S_r, d\sigma)}^k \right)}.$$

This implies the conclusion of Lemma 4.1 and completes the proof. \square

We are ready to prove our main results.

4.1. Proof of Theorem 1.3. In this subsection, we restate Theorem 1.3 and provide a complete proof. The statement of Theorem 1.3 will be a direct consequence from Lemma 4.1 and Lemma 3.3.

Theorem 1.3. *Let $E \subset \mathbb{F}_q^d$. Suppose that $C > 1$ is a sufficiently large constant independent of q .*

(1) *If $d = 4$ or 6 , and $|E| \geq Cq^{\frac{d+1}{2} - \frac{1}{6d+2}}$, then $|\Delta_3(E)| \gtrsim q$.*

(2) *If $d \geq 4$ is even, $k \geq 4$ is an integer, and $|E| \geq Cq^{\frac{d+1}{2} - \frac{1}{6d+2}}$, then $|\Delta_k(E)| \gtrsim q$.*

Proof. We shall prove the statements (1) and (2) of Theorem 1.3 at one time. To the end, notice that if we take $k = 3$ for $d = 4$ or 6 , or if we choose any integer $k \geq 4$ for $d \geq 4$ even, then $k > (12d - 8)/(3d + 4)$ which is the hypothesis of Lemma 3.3. In either case, we therefore invoke the conclusion of Lemma 3.3. In particular, we can take f in Lemma 3.3 to be the characteristic function on a set $E \subset \mathbb{F}_q^d$ so that we have

$$\|\tilde{E}\|_{L^k(S_t, d\sigma)} \lesssim \|E\|_{L^{\frac{k(3d+4)}{(3k-3)d+4k+2}}(\mathbb{F}_q^d, dm)} = |E|^{\frac{(3k-3)d+4k+2}{k(3d+4)}} \quad \text{for all } t \in \mathbb{F}_q^*.$$

Since the constant in the above inequality is independent of $t \in \mathbb{F}_q^*$ and $E \subset \mathbb{F}_q^d$, we see that

$$(4.3) \quad \max_{r \in \mathbb{F}_q^*} \|\tilde{E}\|_{L^k(S_r, d\sigma)}^k \lesssim |E|^{\frac{(3k-3)d+4k+2}{(3d+4)}} \quad \text{for all } E \subset \mathbb{F}_q^d.$$

On the other hand, if $E \subset \mathbb{F}_q^d$ and $|E| \geq 3q^{d/2}$, then it follows from Lemma 4.1 that

$$|\Delta_k(E)| \gtrsim \min \left(q, \frac{|E|^{k+1}}{q^{d-1} \left(\max_{r \in \mathbb{F}_q^*} \|\tilde{E}\|_{L^k(S_r, d\sigma)}^k \right)} \right).$$

Putting this together with (4.3), we obtain that if $|E| \geq 3q^{d/2}$, then

$$|\Delta_k(E)| \gtrsim \min \left(q, \frac{|E|^{k+1}}{q^{d-1} |E|^{\frac{(3k-3)d+4k+2}{3d+4}}} \right).$$

By a direct computation, this implies that there exists a large constant $C > 1$ such that if $|E| \geq Cq^{d(3d+4)/(6d+2)} = Cq^{(d+1)/2 - 1/(6d+2)}$, then $|\Delta_k(E)| \gtrsim q$. Thus, the proof is complete. \square

4.2. Proof of Theorem 1.4. The proof of Theorem 1.4 can be completed by applying Lemma 4.1 and Lemma 3.4. Here, we restate Theorem 1.4 and provide a complete proof.

Theorem 1.4 *Suppose that $d \geq 8$ is even and $E \subset \mathbb{F}_q^d$. Then given $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that if $|E| \geq C_\varepsilon q^{\frac{d+1}{2} - \frac{1}{9d-18} + \varepsilon}$, then $|\Delta_3(E)| \gtrsim q$.*

Proof. Suppose that $d \geq 8$ is even and $E \subset \mathbb{F}_q^d$ with $|E| \geq 3q^{d/2}$. Then Lemma 4.1 with $k = 3$ yields

$$(4.4) \quad |\Delta_3(E)| \gtrsim \min \left(q, \frac{|E|^4}{q^{d-1} \left(\max_{r \in \mathbb{F}_q^*} \|\tilde{E}\|_{L^3(S_r, d\sigma)}^3 \right)} \right),$$

and Lemma 3.4 implies that

$$\max_{r \in \mathbb{F}_q^*} \|\tilde{E}\|_{L^3(S_r, d\sigma)}^3 \lesssim q^{\frac{-3d^2+23d-20}{12d-32}} |E|^{\frac{15d-46}{6d-16}}.$$

Given $\varepsilon > 0$, let $\delta = \varepsilon(9d - 18)/(6d - 16) > 0$. Choose $C_\delta > 0$ such that

$$\max_{r \in \mathbb{F}_q^*} \|\tilde{E}\|_{L^3(S_r, d\sigma)}^3 \leq C_\delta q^\delta q^{\frac{-3d^2+23d-20}{12d-32}} |E|^{\frac{15d-46}{6d-16}}.$$

It follows from this inequality and (4.4) that if $|E| \geq 3q^{d/2}$, then

$$|\Delta_3(E)| \gtrsim \min \left(q, \frac{|E|^4}{q^{d-1} C_\delta q^\delta q^{\frac{-3d^2+23d-20}{12d-32}} |E|^{\frac{15d-46}{6d-16}}} \right).$$

We may assume that $C_\delta > 0$ is a sufficiently large constant. Thus, a direction calculation shows that if

$$|E| \geq C_\delta^{(6d-16)/(9d-18)} q^{\delta(6d-16)/(9d-18)} q^{(9d^2-9d-20)/(18d-36)},$$

then we have $|\Delta_3(E)| \gtrsim q$. Letting $C_\varepsilon = C_\delta^{(6d-16)/(9d-18)}$, we conclude that if

$$|E| \geq C_\varepsilon q^\varepsilon q^{(9d^2-9d-20)/(18d-36)} = C_\varepsilon q^{(d+1)/2-1/(9d-18)+\varepsilon},$$

then $|\Delta_3(E)| \gtrsim q$. This completes the proof. \square

REFERENCES

- [1] J. Bourgain, N. Katz, T. Tao, *A sum-product estimate in finite fields, and applications* Geom. Funct. Anal. **14** (2004), 27–57. [1](#)
- [2] J. Chapman, M. Erdoĝan, D. Hart, A. Iosevich, and D. Koh, *Pinned distance sets, Wolff’s exponent in finite fields and sum-product estimates*, Math.Z. **271** (2012), 63–93. [1](#), [8](#)
- [3] D. Covert, *Results on the Erdős-Falconer distance problem in \mathbb{Z}_q^d for odd q* , preprint (2013), arXiv:1309.1495. [2](#)
- [4] D. Covert, A. Iosevich, and J. Pakianathan, *Geometric configurations in the ring of integers modulo p^ℓ* , to appear, Indiana University Mathematics Journal (2013). [2](#)
- [5] R. Dietmann, *On the Erdős-Falconer distance problem for two sets of different size in vector spaces over finite fields*, Monatshefte für Mathematik, Volume 170 (2013), no. 3-4, 343–359. [2](#)
- [6] D. Hart, A. Iosevich, D. Koh and M. Rudnev, *Averages over hyperplanes, sum-product theory in vector spaces over finite fields and the Erdős-Falconer distance conjecture*, Trans. Amer. Math. Soc. Volume 363, Number 6, (2011), 32553275. [1](#)
- [7] A. Iosevich and D. Koh, *The Erdős-Falconer distance problem, exponential sums, and Fourier analytic approach to incidence theorems in vector spaces over finite fields*, SIAM Journal of Discrete Mathematics, Vol 23, no 1 (2008), 123–135. [2](#)
- [8] A. Iosevich and D. Koh, *Extension theorems for the Fourier transform associated with non-degenerate quadratic surfaces in vector spaces over finite fields*, Illinois J. of Mathematics, Volume 52, Number 2, Summer (2008), 611–628. [8](#)
- [9] A. Iosevich and D. Koh, *Extension theorems for paraboloids in the finite field setting*, Math. Z. **266** (2010), 471–487. [8](#)
- [10] A. Iosevich and D. Koh, *Extension theorems for spheres in the finite field setting*, Forum Math. **22** (2010), no.3, 457–483. [4](#), [8](#)
- [11] H. Iwaniec and E. Kowalski, *Analytic number theory*, AMS Colloquium Publications, **53**, (2004). [5](#)
- [12] A. Iosevich and M. Rudnev, *Erdős distance problem in vector spaces over finite fields*, Trans. Amer. Math. Soc. **359** (2007), 6127–6142. [1](#), [2](#)
- [13] D. Koh and Y. Pi, *Size of dot product sets determined by pairs of subsets of vector spaces over finite fields*, Journal of the Chungcheong Mathematical Society, Volume 26(2013), no. 4, 853–867. [2](#)
- [14] D. Koh and C. Shen, *Sharp extension theorems and Falconer distance problems for algebraic curves in two dimensional vector spaces over finite fields*, Revista Matematica Iberoamericana, Volume 28(2012), no.1, 157–178. [2](#)
- [15] D. Koh and C. Shen, *The generalized Erdős-Falconer distance problems in vector spaces over finite fields*, Journal of Number Theory **132**(2012),no.11, 2455–2473. [2](#)

- [16] D. Koh and H. Sun, *Distance sets of two subsets of vector spaces over finite fields*, PAMS, to appear, arXiv:1212.5305. [2](#), [5](#)
- [17] R. Lidl and H. Niederreiter, *Finite fields*, Cambridge University Press, (1997). [4](#), [5](#)
- [18] M. Lewko, *New restriction estimates for the 3-d paraboloid over finite fields*, arXiv:1302.6664. [8](#)
- [19] M. Lewko, *Finite field restriction estimates based on Kakeya maximal operator estimates*, arXiv:1401.8011. [8](#)
- [20] A. Lewko and M. Lewko, *Endpoint restriction estimates for the paraboloid over finite fields*, Proc. Amer. Math. Soc. 140 (2012), 2013-2028. [8](#)
- [21] G. Mockenhaupt, and T. Tao, *Restriction and Kakeya phenomena for finite fields*, Duke Math. J. **121** (2004), no. 1, 35–74. [8](#)
- [22] I. Shparlinski, *On the set of distance between two sets over finite fields*, International Journal of Mathematics and Mathematical Sciences Volume 2006, Article ID 59482, Pages 1–5. [2](#)
- [23] L. A. Vinh, *Explicit Ramsey graphs and Erdős distance problem over finite Euclidean and non-Euclidean spaces*, Electronic Journal of Combinatorics, 15(1), (2008.). [2](#)

DEPT. OF MATHEMATICS AND COMPUTER SCIENCE, UNIVERSITY OF MISSOURI-ST. LOUIS, ST. LOUIS, MO 63121 USA

E-mail address: `covertdj@umsl.edu`

DEPARTMENT OF MATHEMATICS, CHUNGBUK NATIONAL UNIVERSITY, CHEONGJU CITY, CHUNGBUK-DO 361-763 KOREA

E-mail address: `koh131@chungbuk.ac.kr`

DEPARTMENT OF MATHEMATICS, CHUNGBUK NATIONAL UNIVERSITY, CHEONGJU CITY, CHUNGBUK-DO 361-763 KOREA

E-mail address: `pi@chungbuk.ac.kr`